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ON THE CENTRIFUGAL SEPARATION OF A BULK MIXTURE

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Abstract—The centrifugal separation of a mixture of particles and fluid in an axisymmetric container is examined. The flow consists of three distinct regions—mixture, sediment and purified fluid—with Ekman boundary layers at the interfaces and walls. In the settling process, the mixture and pure fluid acquire retrograde and prograde rotations relative to the tank. This flow pattern, and the shape and locus of the interface which are easily determined, provide another simple means to compare mixture theory and experiment. It is shown that when the Coriolis force is important, the pure fluid layer on the "outwardly" inclined wall is not thin. Moreover the interface between the mixture and the pure fluid is not perpendicular to the centrifugal force. Both features contrast those of the gravitational Boycott effect. As a consequence, there is no obvious enhancement of settling due to geometrical configuration.

1. INTRODUCTION

A rotating axisymmetric container of rather arbitrary shape, figure 1, is filled with a homogeneous mixture of fluid and particles which constitute a volume fraction α_i. The system is assumed initially to be in a state of solid body rotation. Subsequently the centrifugal force causes the mixture to separate into rigidly rotating components of purified fluid and particle sediment of volume fraction α_M.

A much simplified analysis of this problem, Greenspan & Ungarish (1985), was based on certain plausible assumptions about the viscous boundary layers. The major conclusion was that when the Coriolis force is important features which typify the gravitational Boycott effect do not occur, i.e. geometrical shape cannot then enhance separation. Here a much more stringent examination of the Ekman layers is undertaken which in the main supports the earlier findings but also leads to some unexpected and experimentally verifiable results. This makes possible another assessment of the applicability and accuracy of the mixture theory in circumstances for which it was not specifically designed.

2. FORMULATION

It is assumed, and confirmed *a posteriori*, that the fluid at any time is partitioned into three regions: the mixture surrounded by a sediment layer adjacent to one wall and a zone of pure fluid at the other, figure 2. This reflects the fact that a solid particle will either "fall" to or from a solid boundary wherever the local centrifugal force is not tangent to the wall of the container.

Solely for reasons of algebraic simplicity, the sediment layer is taken here to be negligibly thin (which implies a fairly dilute mixture). However, the variable widths of the other two regions are sufficiently large most of the time for the corresponding Ekman number of each, $E = \nu/\Omega H^2$, to be very small (here Ω is the angular velocity of the container, H is the width of the layer and ν is the kinematic viscosity). In essence, this means that boundary layer theory is applicable because the principal effects of shear at walls and interfaces occur only in the Ekman layers. Some of the other assumptions and approximations are:

1. Gravity is neglected compared to the centrifugal force.
2. The particle and fluid densities are nearly the same.
3. The mixture is a Newtonian fluid with an effective viscosity that depends only on the volume fraction α.
4. Ekman layers on all surfaces form instantly.
5. The Ekman layer thickness is much larger than the diameter of a particle.

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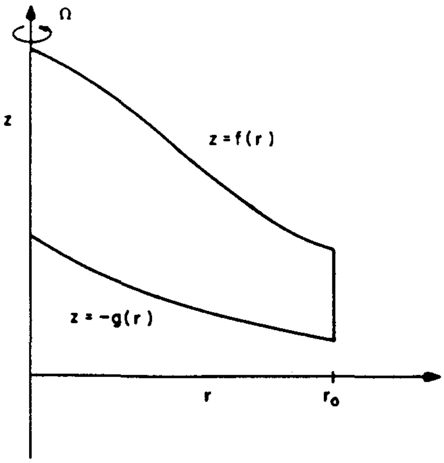

Figure 1. Cross section of an axisymmetric container.

A dimensionless notation is used in which the scales for distance, velocity and time are, respectively, the radius of the container, r_0; the Stokes settling velocity in a centrifugal force field, $U^* = (2/9)(a^2/\nu_C)([\rho_D - \rho_C]/\rho_C)\Omega^2 r_0$; and the settling time r_0/U^*. Subscripts C, D refer to the continuous and discrete phases, a is the radius of the dispersed particle and ρ the density.

The dimensionless parameters that appear are the density ratio

$$\epsilon = \frac{\rho_D - \rho_C}{\rho_C}; \qquad [2.1]$$

the Ekman number

$$E = \frac{\nu_c}{\Omega r_0^2}; \qquad [2.2]$$

the particle Taylor number,

$$\beta = \frac{2}{9}\frac{\Omega a^2}{\nu_c} \qquad [2.3]$$

which measures the particle radius divided by the thickness of the Ekman layer and is assumedly, but not necessarily, small. The equations of mixture theory governing mass and

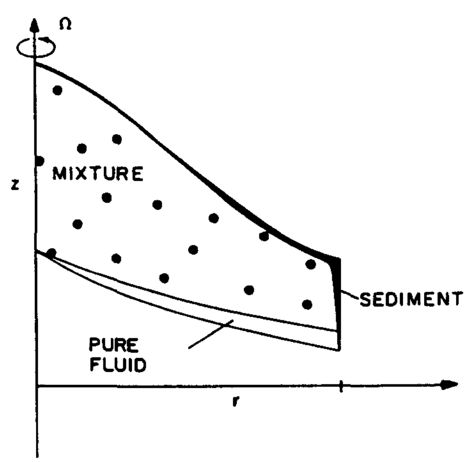

Figure 2. Separation of a mixture showing three regions pure fluid, mixture and sediment, with $\rho_D > \rho_C$.

momentum conservation for incompressible constituents are then:

$$\frac{\partial \alpha}{\partial t} + \nabla \cdot \mathbf{j}_D = 0, \qquad [2.4]$$

$$\nabla \cdot \mathbf{j} = 0, \qquad [2.5]$$

where the particle and volume fluxes are given by

$$\mathbf{j}_D = \alpha \mathbf{q}_D = \alpha \mathbf{q} + \frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\mathbf{q}_R, \qquad [2.6]$$

$$\mathbf{j} = \alpha \mathbf{q}_D + (1-\alpha)\mathbf{q}_C = \mathbf{q} - \frac{\epsilon\alpha(1-\alpha)}{1+\epsilon\alpha}\mathbf{q}_R; \qquad [2.7]$$

$$(1+\epsilon\alpha)\left[2\hat{k}\times\mathbf{q} + |\epsilon|\beta\left(\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\nabla\mathbf{q}\cdot\mathbf{q} + (\nabla\times\mathbf{q})\times\mathbf{q}\right)\right]$$
$$= -\nabla p + \frac{s}{\beta}\alpha r\hat{r} + E\nabla\cdot\underset{\approx}{\pi} - |\epsilon|\beta\nabla\cdot\frac{\alpha(1-\alpha)(1+\epsilon)}{1+\epsilon\alpha}\mathbf{q}_R\mathbf{q}_R, \qquad [2.8]$$

Here $\mathbf{q}$ is the mass velocity of the mixture measured in the cylindrical coordinate system rotating with $\Omega\,\hat{k}$ and $s = \epsilon/|\epsilon|$. The reduced pressure p includes that part of the centrifugal force expressible as gradient. The constitutive laws adopted for the relative velocity $\mathbf{q}_R$ and stress $\underset{\approx}{\pi}$ are:

$$\mathbf{q}_R = \mathbf{q}_D - \mathbf{q}_C = s\,D(\alpha)r\hat{r} \qquad [2.9]$$

where $D(\alpha)$ is an empirically established rule, as for example that given by Ishii & Chawla (1979)

$$D(\alpha) = (1-\alpha)\left(1 - \frac{\alpha}{\alpha_M}\right)^{2.5\alpha_M}; \qquad [2.10]$$

$$\underset{\approx}{\pi} = \mu(\alpha)(\nabla\mathbf{q} + (\nabla\mathbf{q})^+) + \Lambda(\alpha)\nabla\cdot\mathbf{q}\,\underset{\approx}{I}, \qquad [2.11]$$

and

$$\mu(\alpha) = \frac{(1-\alpha)}{D(\alpha)}. \qquad [2.12]$$

(It will not be necessary to specify $\Lambda(\alpha)$; this term could also be included in the definition of the mixture pressure.)

The boundary condition at the wall bordering purified fluid where $\alpha = 0$, is simply $\mathbf{q} \equiv 0$. Since the extremely viscous sediment layer is assumed to be negligibly thin, it is in effect just a wall coating. To good approximation then, the no-slip condition applies to the flow in the mixture layer at the wall where sediment accumulates as well as the normal flux condition $\mathbf{j}\cdot\hat{n} = 0$. The join conditions at the interface

$$z = S(r, t) \qquad [2.13]$$

that separates the pure fluid from the mixture are the continuity of mass flux, tangential stress and normal pressure. These derive from the conservation of mass and momentum across the surface, which is actually a kinematic shock, and the no-slip condition applied there to the velocity, $\mathbf{q}$. The relevant equations are given later in order to take the boundary layer approximation into account.

3. ANALYSIS: SLOW SETTLING

Consider the separation of an almost neutrally buoyant mixture for which the parameter ϵ is very small. For definiteness take ϵ to be positive so that $\rho_D > \rho_C$, and let H be the typical height of the container with $H/r_0 = 0(1)$. Since $\lambda = E^{1/2}/H\epsilon\beta$ is the ratio of the separation and spin-up times, $1 \ll \lambda$ means that separation is a relatively slow and long process. Similarly, with $\epsilon = cE^{1/2}$ comparatively fast or slow separative processes can be studied by setting the magnitude of c, or $c\beta$. Only the case $c = 0$ is presented here in any detail although some results for nonzero c are quoted later.

Perturbation expansions of the basic equations of motion in powers of ϵ yield as the lowest order nonlinear theory

$$\nabla \cdot \mathbf{q} = 0, \qquad [3.1]$$

$$2\hat{k} \times \mathbf{q} = -\nabla p + \frac{1}{\beta}\alpha r\hat{r} - E\mu(\alpha)\nabla \times \nabla \times \mathbf{q}, \qquad [3.2]$$

$$\alpha_t + \nabla \cdot (\alpha\mathbf{q} + \alpha(1-\alpha)\mathbf{q}_R) = 0, \qquad [3.3]$$

with

$$\mathbf{q}_R = r\, D(\alpha)\hat{r}. \qquad [3.4]$$

The initial state is one of uniform and constant volume fraction, $\alpha(0) = \alpha_i$, and it follows from [3.3] that along characteristic paths

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -2\alpha(1-\alpha)D(\alpha). \qquad [3.5]$$

Thus the volume fraction *in the mixture region* is everywhere a function of time only

$$\alpha = \alpha(t) \qquad [3.6]$$

(which for a dilute mixture is $\alpha \simeq \alpha_i e^{-2t}$). This is an important simplification because it allows the remaining equations to be solved using the standard boundary layer theory of rotating fluids. In this procedure, the Ekman layers at each wall and on either side of the interface are analyzed to determine the equivalent boundary conditions that apply to the "inviscid" motion in regions bordered by the viscous layers. (Note that [3.6] justifies the form of the shear terms in [3.2].)

According to [3.5], the volume fraction α is a constant or a known function of time only, in each of the different regions. The same is obviously true of the viscosity coefficient $\mu(\alpha)$ in [3.2] where the now conservative buoyancy force, $(1/\beta)\alpha r\hat{r} = \nabla(\alpha/2\beta)r^2$, may be combined with the pressure gradient. Since time is essentially a parameter, the basic equations [3.1] and [3.2] are exactly those governing the flow of incompressible rotating fluids. The results of that theory, see Greenspan (1968) especially section 2.17, are directly applicable if proper care is exercised to account for the different viscosities of the mixture and purified fluid. We make use of the formula for the normal velocity of the inviscid, interior flow at a surface with a tangential velocity $\mathbf{V}_w$ and at which there is a slow normal flux of magnitude $E^{1/2}M$:

$$\mathbf{q} \cdot \hat{n} = -E^{1/2}\left[M - \frac{1}{2}\hat{n} \cdot \nabla \times \left(\{\hat{n} \times (\mathbf{q} - \mathbf{V}_w) + \frac{\hat{n} \cdot \hat{k}}{|\hat{n} \cdot \hat{k}|}(\mathbf{q} - \mathbf{V}_w)\}\,|\hat{n} \cdot \hat{k}|^{-1/2}\right)\right]. \qquad [3.7]$$

Here $\hat{n}$ is the unit normal to the boundary which points out of the fluid and E must be interpreted as necessary to include the factor $\mu(\alpha)$. The interface between mixture and purified fluid is viewed as such a "wall" and the unknown values of M and $\mathbf{V}_w$ are determined to satisfy the join conditions there which relate to the conservation of mass and momentum and the no-slip velocity constraint. (Since ϵ is small, the mass-averaged velocity $\mathbf{q}$ is continuous across the shock but $\mathbf{q}_C$ and $\mathbf{q}_D$ are not and in general change values from one side to the other.)

From the analysis of the boundary layer, the continuity of total pressure and tangential shear stress across the interface imply that

$$\mathbf{q}_{\mathrm{I}} - \mathbf{q}_{\mathrm{II}} = -\frac{\alpha r}{2\beta}\hat{\theta}, \qquad [3.8]$$

and

$$\mu^{1/2}(\mathbf{q}_{\mathrm{I}} - \mathbf{V}_w) + (\mathbf{q}_{\mathrm{II}} - \mathbf{V}_w) \equiv 0, \qquad [3.9]$$

where I, II designate variables in the mixture and the purified fluid. Figure 3, a cross section of the container, shows these regions and the (four) Ekman layers at the walls and on either side of the interface $z = S(r, t)$. (The sediment layer is the top wall.)

The inviscid interior equations, [3.1], [3.2] with $E = 0$ can now be solved in each region and the solutions joined properly across the kinematic shock. The motion of this shock perpendicular to itself is examined in the next section. The effects of tangential and normal movement can be considered separately because "steady" Ekman layers were assumed to form instantly on all surfaces. In terms of the geometrical factors

$$N = (1 + S_r^2)^{1/2}, \quad N_T = [1 + (f'(r))^2]^{1/2}, \quad N_B = [1 + (g'(r))^2]^{1/2}, \qquad [3.10]$$

and normal vectors, $\mathbf{n} = S_r\hat{r} - \hat{k}$, $\mathbf{n}_T = -f'(r)\hat{r} + \hat{k}$, $\mathbf{n}_B = -g'(r)\hat{r} - \hat{k}$ the following results are obtained. The velocity components, in cylindrical coordinates, are

$$u_{\mathrm{I}} = -u_{\mathrm{II}} = 0, \qquad [3.11]$$

$$v_{\mathrm{I}} = -\frac{r\alpha}{2\beta}\frac{N_B^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}}, \qquad v_{\mathrm{II}} = \frac{r\alpha}{2\beta}\frac{\mu^{1/2}N_T^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}}, \qquad [3.12]$$

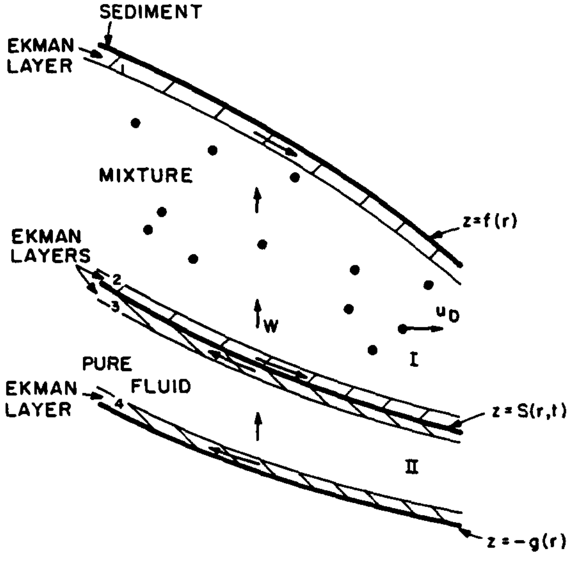

Figure 3. The Ekman layers at walls and interfaces showing directions of flow and mass transport.

$$w_{\mathrm{I}} = w_{\mathrm{II}} = \frac{E^{1/2}\mu^{1/2}\alpha}{4\beta r}\frac{\partial}{\partial r}\left(\frac{r^2(N_T N_B)^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}}\right). \qquad [3.13]$$

Relative to the container, *the regions of the mixture and purified fluid have retrograde and prograde rotations*, respectively.

Calculations of the normalized outward mass transports in the four Ekman layers $\bar{Q} = Q/2\pi r E^{1/2}$ yield

$$\bar{Q}_1 = -\frac{\mu^{1/2}N_T^{1/2}v_{\mathrm{I}}}{2}, \quad \bar{Q}_2 = \frac{\mu^{1/2}N^{1/2}r\alpha}{4\beta(\mu^{1/2}+1)} = -\bar{Q}_3, \quad \bar{Q}_4 = -\frac{N_B^{1/2}v_{\mathrm{II}}}{2}, \qquad [3.14]$$

and the flux at the shock $z = S(r, t)$ is given by

$$\mathcal{S} = E^{1/2}M = -\mathbf{q}\cdot\hat{n} = \frac{E^{1/2}\mu^{1/2}\alpha}{4\beta N r}\frac{\partial}{\partial r}\left(\frac{r^2(N_T N_B)^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}} + \frac{r^2 N^{1/2}}{1+\mu^{1/2}}\right). \qquad [3.15]$$

According to [3.9], the interface rotates with velocity

$$\mathbf{V}_w = \frac{\mu^{1/2}v_{\mathrm{I}} + v_{\mathrm{II}}}{\mu^{1/2}+1}\hat{\theta} = \frac{r\alpha}{2\beta}\frac{\mu^{1/2}}{\mu^{1/2}+1}\frac{N_T^{1/2} - N_B^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}}\hat{\theta}. \qquad [3.16]$$

The vertical component of velocity is directed from the purified fluid to the mixture and were the interface itself not moving slowly, the Ekman layers would actually tend to *mix* the fluid. This implies that centrifugal separation in steady-state processing involving fluid layers could not occur if the rotational boundary layers in *every* region were exceptionally thin. The boundary layer assumptions must then be invalid somewhare in such devices, in at least one of the liquid regions where shear stresses are important thoughout. There the Ekman layers must actually overlap to an extent so that shear counteracts the Coriolis force. The diminution of the Coriolis force by shear stresses, or physical barriers Greenspan & Ungarish (1984), is important for efficient separation. Certain centrifuges utilize many very closely spaced plates so that the Ekman number based on gap height is never "large" in an asymptotic sense. Analysis of separating flows in such configurations, Carlsson (1979), treats the full equations of motion in circumstances when the boundary layers in one domain substantially overlap or even merge into a Couette-type flow.

Since $u = 0(\epsilon)$ and

$$u_D = u + \frac{1-\alpha}{1+\epsilon\alpha}u_R \qquad [3.17]$$

it follows that $u_D = u_R + 0(\epsilon)$. This shows that there is very little enhancement of settling velocity caused by geometry and implies that the pure fluid layer on the wall from which particles are removed does not remain thin during separation. When the Coriolis force is dominant, there is essentially no Boycott effect in a centrifigual force field like that observed in ordinary gravitational settling.

4. MOTION OF THE INTERFACE

The interface between purified fluid and mixture

$$z = S(r, t) \qquad [4.1]$$

consists of those particles that were originally on the wall of the container. (The particle is actually an infinitesimal element of the dispersed phase continuum.) Therefore

$$w_D = S_r u_D + S_t \qquad [4.2]$$

or equivalently

$$S_t + \mathbf{n} \cdot \mathbf{q}_D = 0 \qquad [4.3]$$

where

$$\mathbf{n} = S_r\hat{r} - \hat{k} = N\hat{n} = (1 + S_r^2)^{1/2}\hat{n}. \qquad [4.4]$$

Obviously, the velocity of this surface is the same as the particle velocity of the dispersed phase, $\mathbf{U} = \mathbf{q}_D$. In particular the position of the surface is determined by the normal component of velocity

$$\mathcal{U} = \hat{n} \cdot \mathbf{U} = \hat{n} \cdot \mathbf{q}_D, \qquad [4.5]$$

which is calculated from the kinematic shock conditions across S. Conservation of mass requires that the normal fluxes on the plus and minus sides of the discontinuity be related by

$$(1 - \alpha)(\mathbf{q}_c - \mathbf{U}) \cdot \hat{n}]_-^+ = 0, \qquad \alpha(\mathbf{q}_D - \mathbf{U}) \cdot \hat{n}]_-^+ = 0,$$

and

$$\mathbf{j} \cdot \hat{n}]_-^+ = 0$$

where $\mathbf{j}$ is defined in [2.7]. Since by definition

$$\mathbf{q}_D = \mathbf{q} + (1 - \alpha)rD(\alpha)\hat{r}, \qquad \mathbf{q}_C = \mathbf{q} - \alpha rD(\alpha)\hat{r},$$

these relations, with that for the normal flux across the interface given by [3.15]

$$\mathbf{q} \cdot \hat{n} = -\mathcal{S} = -E^{1/2}M,$$

lead to the equation

$$\mathcal{U} = -\mathcal{S} + (1 - \alpha)D(\alpha)r\hat{r} \cdot \hat{n}. \qquad [4.6]$$

In addition, some other important values on the plus and minus sides of the surface are

$$(\mathbf{q}_C \cdot \hat{n})_+ = -\mathcal{S} - \alpha D(\alpha)r\hat{r} \cdot \hat{n}, \qquad (\mathbf{q}_C \cdot \hat{n})_- = -\mathcal{S}, \qquad (\mathbf{q}_D \cdot n)_+ = \mathcal{U}.$$

Relative to the front, fluid crosses the shock and slows.

According to [4.6], geometrical effects do increase the velocity of the interface by the magnitude of the normal flux $\mathcal{S}$. However, since $\mathcal{S} = 0(E^{1/2}\alpha/\beta)$, which has been assumed small in this analysis to apply boundary layer theory, enhancement is likewise small compared to the centrifugal settling speed. Thus, no significant Boycott effect exists in axisymmetric containers, a conclusion consistent with earlier work of Greenspan & Ungarish (1984).

The full equation for the position of the interface is

$$S_t + (1 - \alpha)D(\alpha)rS_r - \frac{E^{1/2}\mu^{1/2}\alpha}{4\beta r}\frac{\partial}{\partial r}r^2\left[\frac{(N_TN_B)^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}} + \frac{N^{1/2}}{1 + \mu^{1/2}}\right] = 0, \qquad [4.7]$$

where

$$\alpha_t = -2\alpha(1 - \alpha)D(\alpha). \qquad [4.8]$$

The surface is essentially the same as that obtained by ignoring the enhancement term in [4.7]. However, as β decreases so does the radial velocity of a particle, although its azimuthal and axial velocity components remain $0(\alpha\epsilon)$ and $0(\alpha\epsilon E^{1/2})$, respectively. Thus as particle size decreases, enhancement due to the axial velocity assumes more importance in [4.7]. It is therefore of interest to test the range of validity of the analysis and to consider small but not negligible values of $E^{1/2}/\beta$ which are certainly attainable experimentally. For the special case of conical boundaries where N_T and N_B are both constants, the preceding equation can be solved exactly by writing

$$S(r,t) = A(t) + rB(t), \qquad [4.9]$$

i.e. the interface is also a cone. It follows that

$$A'(t) = \frac{E^{1/2}\mu^{1/2}}{2\beta}\alpha\left[\frac{(N_T N_B)^{1/2}}{\mu^{1/2}N_T^{1/2} + N_B^{1/2}} + \frac{N^{1/2}}{1+\mu^{1/2}}\right], \qquad [4.10]$$

$$B(t) = B(0)\sqrt{\frac{\alpha(t)}{\alpha(0)}}, \qquad [4.11]$$

with

$$N = [1 + B^2(t)]^{1/2}.$$

Initially the interface and the inner wall of the container are the same and this sets the values of $A(0)$, $B(0)$. Numerical integration of [4.8] and [4.10] for any drag law gives the complete solution. For dilute mixtures $\alpha \simeq \alpha(0)e^{-2t}$ and $\mu(\alpha) \simeq 1$, analytical expressions can be obtained:

$$B(t) = B(0)e^{-t} = be^{-t}$$

$$A(t) = \frac{E^{1/2}\alpha(0)}{4\beta}\left[\frac{(1-e^{-2t})(N_T N_B)^{1/2}}{N_T^{1/2} + N_B^{1/2}} + \frac{2}{5b^2}\{(1+b^2)^{5/4} - (1+b^2e^{-2t})^{5/4}\}\right] + A(0).$$

A slightly, but perhaps measurably faster conical shock is predicted in all cases. The locus of the interface in containers of more general shape can be obtained using perturbation methods.

5. MODERATE SETTLING RATES

A similar analysis can be performed in the parameter range

$$\epsilon = cE^{1/2}$$

with $c = 0(1)$. The principal terms in the radial and axial momentum equations are, as in the previous case,

$$v_{\mathrm{II}} - v_{\mathrm{I}} = \frac{1}{2\beta}\alpha r, \qquad \frac{\partial v_{\mathrm{I}}}{\partial z} = \frac{\partial v_{\mathrm{II}}}{\partial z} = 0, \qquad [5.1]$$

while for the azimuthal balance

$$cE^{1/2}\beta\frac{\partial v_{\mathrm{I}}}{\partial t} + 2u_{\mathrm{I}} = 0, \qquad [5.2]$$

$$cE^{1/2}\beta\frac{\partial v_{\mathrm{II}}}{\partial t} + 2u_{\mathrm{II}} = 0. \qquad [5.3]$$

The treatment of the shear layers poses several difficulties. First of all, nonlinear terms occur in a formal expansion procedure. However, experience with homogeneous rotating fluids indicates that good qualitative results can be obtained by using the equivalent Ekman "suction" boundary conditions as given by the linear theory. Secondly, the difference between mass and volume velocities, $\mathbf{q}$ and $\mathbf{j}$, which is an $0(\epsilon)$ quantity, is no longer negligible. This requires careful reconsideration of the boundary conditions, especially for $\hat{n}\cdot\mathbf{q}$. Moreover, a formula for the relative velocity $\mathbf{q}_R$ in the shear layer is needed. Since the radial pressure gradient is unaffected in these layers, [3.4] seems a reasonable approximation, provided the particles are small, i.e. $\beta \ll 1$. It is assumed that the sediment on $z = f(r)$ is motionless and thin, in which case the boundary conditions there are $\mathbf{j}\cdot\hat{n} = 0$ and $\hat{n}\times(\hat{n}\times\mathbf{q}) = 0$. Finally, the Ekman layer suction at this boundary is [3.7] with

$$E^{1/2}M_T = -\epsilon\alpha(1-\alpha)D(\alpha)r\hat{r}\cdot\hat{n}_T.$$

Although the mass and volume fluxes across the interface $z = S(r, t)$ are obviously continuous functions, the mass velocity is not. This implies that $-E^{1/2}M$ and $-E^{1/2}M + \epsilon\alpha(1-\alpha)D(\alpha)r\hat{r}\cdot\hat{n}$ must be used in [3.7] for the Ekman layers adjoining the interface, i.e. on the pure fluid and mixture sides, respectively.

The volume transport in an Ekman layer is now

$$Q = -\pi r E^{1/2}\,|\,\hat{n}\cdot\hat{k}\,|^{1/2}(v - \mathbf{V}_w\cdot\hat{\theta}), \qquad [5.4]$$

where E is based on the appropriate viscosity and the positive direction is toward the periphery. It follows that, again, $Q_2 + Q_3 = 0$.

An appropriate matching of regions I and II using the total volume flux balance

$$Q_1 + Q_2 + Q_3 + Q_4 + 2\pi r\int_{-g}^{f}\mathbf{j}\cdot\hat{r}\,\mathrm{d}r = 0,$$

yields the equation for the azimuthal motion,

$$Hc\beta\frac{\partial\omega_{\mathrm{I}}}{\partial t} + (\mu^{1/2}N_T^{1/2} + N_B^{1/2})\omega_{\mathrm{I}} = -\frac{1}{2}N_B^{1/2}\alpha + Hc\beta\left(1 - \frac{3H_{\mathrm{I}}}{H}\right)\alpha(1-\alpha)D(\alpha), \qquad [5.5]$$

where $\omega_{\mathrm{I}} = \beta v_{\mathrm{I}}/r$, $H = f + g$, $H_{\mathrm{I}} = f - S$.

The last term couples this equation to that for $S(r, t)$, which is obtained as in section 4 but with the appropriate value of M used:

$$S_t + (1-\alpha)D(\alpha)rS_r - \left(\frac{E^{1/2}}{\beta}\right)\frac{1}{2r}\frac{\partial}{\partial r}r^2\left[\frac{\alpha}{2}\left(\frac{\mu^{1/2}N^{1/2}}{1+\mu^{1/2}} + N_B^{1/2}\right) + N_B^{1/2}\omega_{\mathrm{I}}\right.$$
$$\left. + Hc\beta\left(1 - \frac{H_{\mathrm{I}}}{H}\right)\frac{\partial\omega_{\mathrm{I}}}{\partial t} - Hc\beta\left(1 - \frac{H_{\mathrm{I}}}{H}\right)\alpha(1-\alpha)D(\alpha)\right] = 0. \qquad [5.6]$$

Initially, $S(r, 0) = -g(r)$ and $\omega_{\mathrm{I}} = 0$; all other variables can be obtained from ω and S.

Equation [5.5] clearly points out the importance of the parameter $\lambda = (hc\beta)^{-1} = (E^{1/2}/H\,|\,\epsilon\,|\,\beta)$ which is the ratio of the settling and spin-up times. For λ large the process is dominated by the Ekman layers, while in the opposite case separation prevails. In any event, however, the radial velocity of the mixture in the core is small, $0(\epsilon)$. The equations reduce correctly in the special cases $c = 0$, or $\alpha = 0$ and, for c large, give the correct formula for the residual retrograde rotation in rapid settling. However, this direction will not be pursued further at this time.

6. CONCLUSIONS

Analysis of viscous effects on the centrifugal settling of a uniform batch mixture shows that when the Coriolis force is important, geometry can have only a minor influence in accelerating the process. Conversely, enhanced settling due to changes of geometry, a Boycott effect, is made possible by diminishing the Coriolis force. In practice this is accomplished by using a disk stack with a small gap thickness so that the shear forces are everywhere important, or by blocking the flow with meridional sections, Greenspan & Ungarish (1985).

Some of the theoretical results—the occurrence of Ekman layers and the mass transports within them, the retrograde and prograde rotations of the mixture and pure fluid, and the locus of the interface between mixture and pure fluid—make for a relatively easy and different experimental assessment of the mixture equations.

Mixture theory, in which there are three distinct regions—sediment, mixture and purified fluid—separated by kinematic shocks, is at present a most practical framework for the study of multiphase fluid dynamics in which shear stresses and wall effects are important. The problems are rendered solvable, the results seem consistent with observation and some of the intrinsic difficulties of two phase flow theory are avoided [Carlsson (1979), Probstein *et al.* (1977), Acrivos & Herbolzheimer (1979), Bark & Johansson (1982)].

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APPENDIX

The results for negative buoyancy, $\rho_D \le \rho_C$ are summarized here. The geometry is the same but now the lighter particles move away from the top surface $z = f$, and sediment is formed on the bottom wall $z = -g$, figure 4. Again, subscripts I and II denote the mixture and pure fluid regions.

In this case,

$$\mathbf{q}_R = -r\,D(\alpha)\hat{r}, \qquad [A.1]$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = 2\alpha(1-\alpha)\,D(\alpha), \qquad [A.2]$$

$$v_{\mathrm{I}} - v_{\mathrm{II}} = \frac{1}{2\beta}\alpha r, \qquad [A.3]$$

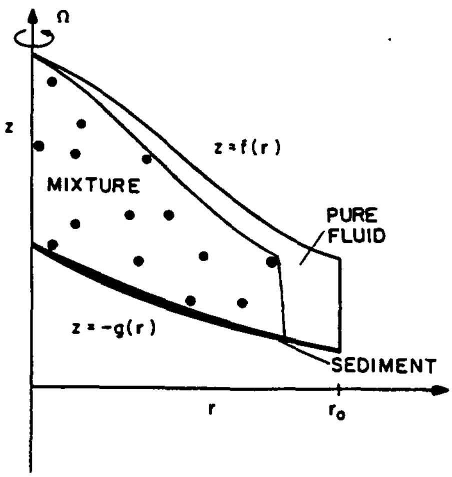

Figure 4. Separation of a mixture of light particles, $\rho_D < \rho_C$.

$$V_W = \mathbf{V}_w \cdot \hat{\theta} = \frac{\mu^{1/2} v_{\mathrm{I}} + v_{\mathrm{II}}}{1 + \mu^{1/2}} \qquad \text{[A.4]}$$

$$v_{\mathrm{II}} - V_w = -\frac{\mu^{1/2}}{1 + \mu^{1/2}} \frac{1}{2\beta} \alpha r, \qquad \text{[A.5]}$$

$$\overline{Q}_2 = -\overline{Q}_3 = \frac{\alpha r}{4\beta} \frac{(\mu N)^{1/2}}{1 + \mu^{1/2}}, \qquad \text{[A.6]}$$

and

$$S_t + (1 - \alpha)\, D(\alpha) r\, S_r - N E^{1/2} M = 0. \qquad \text{[A.7]}$$

Moreover for $c = 0$, with $R = N_T^{1/2} + \mu^{1/2} N_B^{1/2}$, it follows that

$$u_{\mathrm{I}} = u_{\mathrm{II}} = 0 \qquad \text{[A.8]}$$

$$v_{\mathrm{I}} = \frac{\alpha r}{2\beta} \frac{N_T^{1/2}}{R}; \qquad v_{\mathrm{II}} = -\frac{\alpha r}{2\beta} \frac{(\mu N_B)^{1/2}}{R} \qquad \text{[A.9]}$$

$$w_{\mathrm{I}} = w_{\mathrm{II}} = \frac{E^{1/2} \mu^{1/2}}{4\beta r} \alpha \frac{\partial}{\partial r} r^2 \frac{(N_T N_B)^{1/2}}{R} \qquad \text{[A.10]}$$

$$\overline{Q}_1 = -\overline{Q}_4 = \frac{\alpha r}{4\beta} \frac{(\mu N_T N_B)^{1/2}}{R} \qquad \text{[A.11]}$$

$$V_w = \frac{\alpha r}{2\beta} \frac{\mu^{1/2}}{1 + \mu^{1/2}} \frac{N_T^{1/2} - N_B^{1/2}}{R} \qquad \text{[A.12]}$$

and

$$E^{1/2} M = \frac{E^{1/2}}{4\beta} \alpha \frac{\mu^{1/2}}{N} \frac{1}{r} \frac{\partial}{\partial r} r^2 \left[\frac{N^{1/2}}{1 + \mu^{1/2}} + \frac{(N_T N_B)^{1/2}}{R} \right]. \qquad \text{[A.13]}$$